Lecture 8: Fiber homotopy and homotopy fiber

Path space

Definition

Given a space $X \in \underline{\mathscr{T}}$, and $x \in X$, we define

$$\text{free path space}: \quad PX = \mathrm{Map}(I, X)$$

$$\text{based path space} \quad : \quad P_xX = \mathrm{Map}((I, 0), (X, x)).$$

We denote the two maps

$$\begin{tikzcd} & PX \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ X & & X \end{tikzcd}$$

where $p_0(\gamma) = \gamma(0)$ is the start point and $p_1(\gamma) = \gamma(1)$ is the end point of the path γ. It induces

$$p = (p_0, p_1) : PX \to X \times X.$$

Theorem

Let $X \in \mathscr{T}$. Then

1. $p : PX \to X \times X$ is a fibration.
2. The map $p_0 : PX \to X$ is a fibration whose fiber at x_0 is $P_{x_0}X$.
3. The map $p_1 : P_{x_0}X \to X$ is a fibration whose fiber at x_0 is $\Omega_{x_0}X$.
4. $p_0 : PX \to X$ is homotopy equivalence. $P_{x_0}X$ is contractible.

Proof

(1) We need to prove the HLP of the diagram

$$\begin{tikzcd} Y \times \{0\} \arrow[r] \arrow[d, hook] & PX \arrow[d, "p"] \\ Y \times I \arrow[r] \arrow[ur, dashed, "?"] & X \times X \end{tikzcd}$$

By the Exponential Law, this is equivalent to the extension problem

$$\begin{tikzcd} Y \times \{0\} \times I \cup Y \times I \times \partial I \arrow[r] \arrow[d] & X \\ Y \times I \times I \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

This follows by observing that $\{0\} \times I \cup I \times \partial I$ is a deformation retract of $I \times I$.

(2) follows from the composition of two fibrations

$$\begin{tikzcd} P_X \arrow[r] \arrow[dr] & X \times X \arrow[d] \\ & X \end{tikzcd}$$

(3) follows from the pull-back diagram and the fact that fibrations are preserved under pull-back

$$\begin{tikzcd} P_{x_0}X \arrow[r] \arrow[d] & PX \arrow[d] \\ X \arrow[r, "x_0 \times \mathrm{id}"] & X \times X. \end{tikzcd}$$

(4) follows from the retracting path trick. □

Definition

Let $f\colon X \to Y$. We define the mapping path space P_f by the pull-back diagram

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] & PY \arrow[d, "p_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

An element of P_f is a pair (x, γ) where

$$\gamma : I \to Y, \quad \gamma(1) = f(x).$$

Let

$$\iota : X \hookrightarrow P_f, \quad x \mapsto (x, 1_{f(x)})$$

and $p : P_f \to Y$ be the start point of the path. We have

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & P_f \arrow[d, "p"] \\ & Y \end{tikzcd}$$

Theorem

$\iota : X \to P_f$ is strong deformation retract (hence homotopy equivalence) and $p : P_f \to Y$ is a fibration. In particular, any f is a composition of a homotopy equivalence with a fibration.

This theorem says every map is equivalent to a fibration in $\mathrm{h}\mathscr{T}$.

Proof

The first statement follows from the retracting path trick. We prove p is a fibration. Consider the pull-back diagram

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] & PY \arrow[d] \\ Y \times X \arrow[r, "\mathrm{id} \times f"] & Y \times Y. \end{tikzcd}$$

This implies $P_f \to Y \times X$ is a fibration. Since $Y \times X \to Y$ is also a fibration, so is the composition

$$p : P_f \to Y \times X \to Y.$$

$\square$

Fiber homotopy

Definition

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fibrations. A fiber map from p_1 to p_2 is a map $f : E_1 \to E_2$ such that $p_1 = p_2 \circ f$.

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Definition

Two fiber maps $f_0, f_1 : p_1 \to p_2$ are said to be fiber homotopic

$$f_0 \simeq_B f_1$$

if there exists a homotopy $F: E_1 \times I \to E_2$ from f_0 to f_1 such that $F(-, t)$ is a fiber map for each $t \in I$.

Definition

$f: p_1 \to p_2$ is a fiber homotopic equivalence if there exists $g: p_2 \to p_1$ such that both $f \circ g$ and $g \circ f$ are fiber homotopic to identity maps.

Proposition

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fibrations and $f \colon E_1 \to E_2$ be a fiber map. Assume $f \colon E_1 \to E_2$ is a homotopy equivalence, then f is a fiber homotopy equivalence. In particular, $f \colon p_1^{-1}(b) \to p_2^{-1}(b)$ is a homotopy equivalence for any $b \in B$.

$$\begin{tikzcd}
E_1 \arrow[rr, bend left, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[ll, dotted, bend left, "g"] \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

Proof

We only need to prove that for any fiber map $f\colon E_1 \to E_2$ which is a homotopy equivalence, there is a fiber map $g : E_2 \to E_1$ such that $g \circ f \simeq_B 1$. In fact, such a g is also a homotopy equivalence and we can find $h : E_1 \to E_2$ such that $h \circ g \simeq_B 1$. Then

$$f \simeq_B h \circ g \circ f \simeq_B h$$

which implies $f \circ g \simeq_B 1$ as well.

Let $g : E_2 \to E_1$ represent the inverse of the homotopy class $[f]$ in $\underline{\mathrm{h}\mathscr{T}}$. We first show that we can choose g to be a fiber map, i.e., $p_1 \circ g = p_2$ in the following diagram

$$\begin{tikzcd} & E_1 \arrow[d, "p_1"] \\ E_2 \arrow[ur, "g"] \arrow[r, "p_1 \circ g"'] & B \end{tikzcd}$$

Otherwise, we observe that $p_1 \circ g = p_2 \circ f \circ g \simeq p_2$. We can use the fibration p_1 to lift the homotopy $p_1 \circ g \simeq p_2$ to a homotopy $g \simeq g'$. Then g' is a fiber map, and we can replace g by g'.

$$
\begin{tikzcd}
 & E_1 \arrow[dd, "p_1"] \\
E_2 \arrow[ur, bend left, "g"] \arrow[ur, bend right, "g'"'] \arrow[ur, phantom, "\Downarrow"] \arrow[dr, bend left, "p_1 \circ g"] \arrow[dr, bend right, "p_2"'] \arrow[dr, phantom, "\Downarrow"] & \\
 & B
\end{tikzcd}
$$

Now we assume $g : E_2 \to E_1$ is a fiber map. The problem can be further reduced to the following

Claim

Let $p : E \to B$ be a fibration and $f : E \to E$ is a fiber map that is homotopic to 1_E, then there is a fiber map $h : E \to E$ such that $h \circ f \simeq_B 1$.

In fact, let $f : E_1 \to E_2$ as in the proposition, $g : E_2 \to E_1$ be a fiber map such that $g \circ f \simeq 1$ as chosen above. The "Claim" implies that we can find a fiber map $h : E_1 \to E_1$ such that $h \circ g \circ f \simeq_B 1$. Then the fiber map $\tilde{g} = h \circ g$ has the required property that $\tilde{g} \circ f \simeq_B 1$.

Now we prove the “Claim”. Let F be a homotopy from f to 1_E and $G = p \circ F$. Since p is fibration, we can construct a homotopy H that starts from 1_E and lifts G. Here is the picture

$$
\begin{tikzcd}
E \arrow[r, bend left, "f"] \arrow[r, bend right, "1_E"'] \arrow[r, phantom, "\Downarrow F"] \arrow[d, equal] & E \arrow[r, "h"] \arrow[d, "p"] & E \arrow[dl, "p"] \\
E \arrow[r, bend left, "p"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow G"] & B &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, bend left, "1_E"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow H"] \arrow[d, equal] & E \arrow[d, "p"] \\
E \arrow[r, bend left, "p"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow G"] & B
\end{tikzcd}
$$

Combining these two homotopies we find a homotopy $\tilde{F}$ from $h \circ f$ to 1_E that lifts the following homotopy

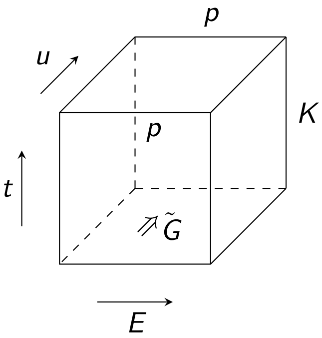
$$\tilde{G} : E \times I \to B, \quad \tilde{G}(-, t) = \begin{cases} G(-.2t) & 0 \le t \le 1/2 \\ G(-, 2-2t) & 1/2 \le t \le 1 \end{cases}$$

Here is the picture

$$\begin{tikzcd}
E \arrow[r, bend left, "h\circ f"] \arrow[r, bend right, "1_E"'] \arrow[r, phantom, "\Downarrow\tilde{F}"] \arrow[d, equal] & E \arrow[d, "p"] \\
E \arrow[r, bend left, "p"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow\tilde{G}"] & B
\end{tikzcd}$$

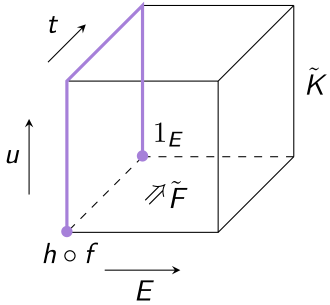
We can construct a map $K : E \times I \times I \to B$ that gives a homotopy between $\tilde{G} : E \times I \to B$ and the projection $E \times I \to E \xrightarrow{p} B$ (by pushing the two copies of G in $\tilde{G}$)

$$\begin{aligned} K(-, u, 0) &= \tilde{G}(-, u), \\ K(-, u, 1) &= p(-), \\ K(-, 0, t) &= p(-), \\ K(-, 1, t) &= p(-), \quad \forall u, t \in I. \end{aligned}$$



Since p is a fibration, we can find a lift $\tilde{K}: E \times I \times I \to E$ of K s.t.

$$\tilde{K}(-, u, 0) = \tilde{F}(-, u).$$



Then we have the following fiber homotopy

$$h \circ f = \tilde{K}(-, 0, 0) \simeq_B \tilde{K}(-, 0, 1) \simeq_B \tilde{K}(-, 1, 1) \simeq_B \tilde{K}(-, 1, 0) = 1_E.$$

$\square$

Homotopy fiber

Definition

Let $f: X \to Y$, we define its homotopy fiber over $y \in Y$ to be the fiber of $P_f \to Y$ over y.

Proposition

If Y is path connected, then all homotopy fibers of $f\colon X \to Y$ are homotopic equivalent.

Proof.

Let $y_1, y_2 \in Y$, and F_1, F_2 be the homotopy fiber over y_1, y_2. Then

$$F_i = \{(x, \gamma) | \gamma : I \to Y, \gamma(0) = y_i, \gamma(1) = f(x)\}.$$

Then composition with a path in Y from y_1 to y_2 gives a homotopy equivalence between F_1, F_2. $\square$

If Y is path connected, we will usually write

$$\begin{tikzcd} F \arrow[r] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

where F denotes the homotopy fiber.

Proposition

If $f\colon X \to Y$ is a fibration, then its homotopy fiber at y is homotopy equivalent to $f^{-1}(y)$.

Proof.

We have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & P_f \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where ι is a homotopy equivalence. Then ι is fiber homotopy equivalence. $\square$

Corollary

Let $f: X \to Y$ be a fibration and Y path connected. Then all fibers of f are homotopy equivalent.

Proof.

Given $y_1, y_2 \in Y$, their fibers $f^{-1}(y_1), f^{-1}(y_2)$ are homotopy equivalent to the corresponding homotopy fibers. The corollary follows since all homotopy fibers are homotopy equilvalent. $\square$

Recall the following useful criterion for fibration.

Theorem

Let $p : E \to B$ with B paracompact Hausdorff. Assume there exists an open cover $\{U_\alpha\}$ of B such that $p^{-1}(U_\alpha) \to U_\alpha$ is a fibration. Then p is a fibration.

Corollary

Let $p : E \to B$ be a fiber bundle with B paracompact Hausdorff. Then p is a fibration.

CW complexes and metric spaces are paracompact.